INDIAN JOURNAL
OF MATHEMATICS
Vol. 64, No. 3, 2022, 401-416

HANKEL DETERMINANTS OF CERTAIN ORDER FOR BOUNDED TURNING FUNCTIONS OF ORDER ALPHA

Biswajit Rath *, K. Sanjay Kumar, D. Vamshee Krishna, Ch. Vijaya Kumar, and N. Vani

(*Received* 1 December 2021)

The paper contains an estimation of the best possible upper bound (UB) for the third, fourth order Hankel determinants, 2-fold and 3-fold symmetric functions associated with bounded turning functions of order $\alpha (0 \leq \alpha < 1)$, for a particular value of the parameter α.

1. Introduction

Let $\mathcal{A}$ represent group of mappings f of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in $\mathcal{U}_d = \{z \in \mathcal{C} : |z| < 1\}$, denotes the open unit disc. Pommerenke [23] characterized the r^{th}-Hankel determinant of order n, for f with $r,\ n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, namely

$$H_{r,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix}. \tag{1.2}$$

The Fekete-Szegö functional is obtained for $r = 2$ and $n = 1$ in (1.2), denoted by $H_{2,1}(f)$. Further, sharp bounds to the functional

* *Corresponding Author*

2010 Mathematics Subject Classification: 30C45, 30C50.

Key words and phrases: upper bound, functionals connected with Hankel determinant, Caratheodóry function.

© 2022 Allahabad Mathematical Society

$|a_2a_4 - a_3^2|$, obtained for $r = 2$ and $n = 2$ in (1.2), called as Hankel determinant of order two, given by

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

In recent years, the research on the estimation of an upper bound (UB) to $|H_{2,2}(f)|$ has been focused by many authors. The exact estimates of $|H_{2,2}(f)|$ for the family of functions namely, bounded turning, starlike and convex, symbolized as $\Re$, S^* and $\mathcal{K}$ respectively fulfilling the conditions $\mathrm{Re}\{f'(z)\} > 0$, $\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > 0$ and $\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0$ in the unit disc $\mathcal{U}_d$, were proved by Janteng et al. [11], [12] derived the bounds as $\frac{4}{9}$, 1, and $\frac{1}{8}$, respectively. Choosing $r = 2$ and $n = p+1$ in (1.2), we obtain Hankel determinant of second order for the p-valent function (see [31]), given by

$$H_{2,p+1}(f) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2.$$

A very few papers have been dedicated to $H_{3,1}(f)$, named as 3^{rd} order Hankel determinant obtained for $r = 3$ and $n = 1$ in (1.2). Babalola [5] is the first one, who tried to estimate an UB for $|H_{3,1}(f)|$ for the classes $\Re$, S^* and $\mathcal{K}$. As a consequence of this paper, many papers containing results associated with the Hankel determinant of order 3 (see [3], [6], [7], [13]-[15], [19]-[21], [24]-[30], [32], [34]) and very few papers on order 4 (see [2], [9]), for specific subsets of holomorphic functions were obtained. For our study in this paper, we choose $H_{3,3}(f)$ and $H_{4,1}(f)$, called as Hankel determinant of third order and fourth order, respectively obtained for the values $r = 3,\ n = 3$ and $r = 4,\ n = 1$ in (1.2), given as follows.

$$H_{3,3}(f) = \begin{vmatrix} a_3 & a_4 & a_5 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{vmatrix}, \tag{1.3}$$

and

$$H_{4,1}(f) = \begin{vmatrix} a_1 = 1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix}. \tag{1.4}$$

Motivated with the results obtained by the authors specified above, in the present paper, we estimate an UB to $|H_{3,3,\alpha}(f)|$ and $|H_{4,1,\alpha}(f)|$ for the bounded turning functions of order α, whose respective class is denoted by $\Re(\alpha)$, where $\alpha \in [0,1)$, defined as follows.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\Re(\alpha), \alpha \in [0,1)$, if it satisfies the condition

$$\operatorname{Re}\left\{f'(z)\right\} > \alpha,\ z \in \mathcal{U}_d. \tag{1.5}$$

In deriving our results, the required sharp estimates specified below, given in the form of lemmas, which holds suitable for functions possessing positive real part.
The collection $\mathcal{P}$, of all functions g, each one called as Caratheodóry function [8] of the form,

$$g(z) = 1 + \sum_{t=1}^{\infty} c_t z^t, \tag{1.6}$$

holomorphic in $\mathcal{U}_d$ and $\operatorname{Re} g(z) > 0$, for $z \in \mathcal{U}_d$.

Lemma 1.2 [10]. *If $g \in \mathcal{P}$, then $|c_i - \mu c_j c_{i-j}| \leq 2$, satisfies for the values $i, j \in \mathbb{N}$, with $i > j$ and $\mu \in [0,1]$.*

Lemma 1.3 [17]. *If $g \in \mathcal{P}$, then $|c_i - c_j c_{i-j}| \leq 2$, holds for the values $i, j \in \mathbb{N}$, with $i > j$.*

Lemma 1.4 [22]. *For $g \in \mathcal{P}$, then $|c_t| \leq 2$, for $t \in \mathbb{N}$, equality occurs for the function $h(z) = \frac{1+z}{1-z}$, $z \in \mathcal{U}_d$.*

Lemma 1.5. *For $g \in \mathcal{P}$, then $|c_m c_n - \lambda c_k c_l| \leq 4$, for $m+n = k+l$ with $m, n, k, l \in \mathbb{N}$, where $\lambda \in [0,1]$.*

Proof. In view of Lemma 1.2, consider

$$c_m c_n - \lambda c_k c_l = c_m c_n - c_{m+n} + c_{m+n} - \lambda c_k c_l$$

Using the condition $m+n=k+l$ in the above expression, we have

$$c_m c_n - \lambda c_k c_l = c_m c_n - c_{m+n} + c_{k+l} - \lambda c_k c_l.$$

Taking modulus on both sides of the above expression and then applying the triangle inequality, we obtain

$$|c_m c_n - \lambda c_k c_l| \leq |c_m c_n - c_{m+n}| + |c_{k+l} - \lambda c_k c_l| \leq 2+2=4.$$

Hence the proof of the lemma. □

LEMMA 1.6 [33]. *For $g \in \mathcal{P}$, then*

$$|c_2 - \lambda c_1^2| \leq \begin{cases} 2-\lambda |c_1|^2, \ \lambda \leq \frac{1}{2} \\ 2-(1-\lambda)|c_1|^2, \ \lambda \geq \frac{1}{2}. \end{cases}$$

To procure our results, we adopt the procedure framed through Libera and Zlotkiewicz [16].

2. Bound of $|H_{3,3,\alpha}(f)|$ for the Set $\Re(\alpha)$

THEOREM 2.1. *If $f \in \Re(\alpha)$ and $\alpha \in [0,1)$, then*

$$|H_{3,3,\alpha}(f)| \leq \left[\frac{22(1-\alpha)^3}{105}\right].$$

Proof. For $f \in \Re(\alpha)$, there exists a holomorphic function $g \in \mathcal{P}$ such that

$$f'(z) - \alpha = (1-\alpha)g(z). \tag{2.1}$$

Using the series representation for f and g in (2.1), a simple calculation gives

$$na_n = (1-\alpha)c_{n-1}. \tag{2.2}$$

Putting the values of $a_j, j \in \{3,4,5,6,7\}$ in the expanded determinant of $H_{3,3,\alpha}$, given in (1.3), it simplifies to

$$\begin{aligned} H_{3,3,\alpha}(f) = & \frac{(1-\alpha)^3}{378000}\big[3600c_2c_4c_6 - 3500c_2c_5^2 - 3375c_3^2c_6 \\ & + 6300c_3c_4c_5 - 3024c_4^3\big]. \end{aligned} \tag{2.3}$$

Grouping the terms in the expression (2.3) in order to apply the lemmas mentioned in this paper, we have

$$\begin{aligned} H_{3,3,\alpha}(f) =& \frac{(1-\alpha)^3}{378000}\{3024c_4[c_3c_5 - c_4^2] + 3276c_5[c_3c_{-}c_2c_5] \\ & + 3375c_2[c_2c_4 - c_3^2] + 225c_2\left[c_4c_6 - \frac{224}{225}c_5^2\right]\}. \end{aligned} \tag{2.4}$$

Taking modulus on both side of the expression (2.4), we obtain

$$\begin{aligned} |H_{3,3,\alpha}(f)| \leq & \frac{(1-\alpha)^3}{378000}\{3024|c_4||c_3c_5 - c_4^2| + 3276|c_5||c_3c_{-}c_2c_5| \\ & + 3375|c_2||c_2c_4 - c_3^2| + 225|c_2|\left|c_4c_6 - \frac{224}{225}c_5^2\right|\}. \end{aligned} \tag{2.5}$$

Upon applying Lemmas 1.4 and 1.5 in (2.5), it yeilds

$$|H_{3,3,\alpha}(f)| \leq \left[\frac{22(1-\alpha)^3}{105}\right]. \tag{2.6}$$

Hence the proof of our theorem. $\square$

3. Bound of $|H_{4,1,\alpha}(f)|$ for the Set $\Re(\alpha)$

Theorem 3.1. *If $f \in \Re(\alpha)$ and $\alpha \in [0,1)$, then*

$$|H_{4,1,\alpha}(f)| \leq \begin{cases} \dfrac{(1-\alpha)^3(-17977\alpha + 47250)}{94500}, & \alpha \in \left[0, \dfrac{1827}{12800}\right] \cup \left[\dfrac{10973}{12800}, 1\right) \\[2ex] \dfrac{(1-\alpha)^3 \begin{pmatrix} -460211200\alpha^2 \\ +3337929 \\ +1162828800\alpha \end{pmatrix}}{2419200000\alpha}, & \alpha \in \left[\dfrac{1827}{12800}, \dfrac{1}{2}\right] \\[2ex] \dfrac{(1-\alpha)^2 \begin{pmatrix} 460211200\alpha^2 \\ -1623040000\alpha \\ +1166166729 \end{pmatrix}}{2419200000}, & \alpha \in \left[\dfrac{1}{2}, \dfrac{10973}{12800}\right] \end{cases}.$$

Proof. For $f \in \Re(\alpha)$, using the values of $a_j, j \in \{3,4,5,6,7\}$ in the expande determinant $H_{4,1,\alpha}(f)$ given in (1.4), it yeilds

$$\begin{aligned}
H_{4,1,\alpha}(f) = \frac{(1-\alpha)^3}{6048000}\big[&23625(1-\alpha)c_3^4 - 75600(1-\alpha)c_2c_4c_3^2\\
&- 54000c_6c_3^2 - 63000(1-\alpha)c_5c_3^2c_1\\
&+ 100800c_4c_5c_3 + 56000(1-\alpha)c_2^2c_5c_3\\
&+ 72000(1-\alpha)c_2c_6c_3c_1 + 60480(1-\alpha)c_4^2c_3c_1\\
&- 48384c_4^3 + 26880(1-\alpha)c_2^2c_4^2 + 57600c_2c_4c_6\\
&- 32000(1-\alpha)c_2^3c_6 - 43200(1-\alpha)c_4c_6c_1^2\\
&- 56000c_2c_5^2 + 42000(1-\alpha)c_5^2c_1^2\\
&- 67200(1-\alpha)c_2c_4c_5c_1\big]. \qquad (3.1)
\end{aligned}$$

Rearranging, the terms in (3.1), to apply lemmas mentioned in this paper, we have

$$\begin{aligned}
H_{4,1,\alpha}(f) = \frac{(1-\alpha)^3}{6048000}\big[&32000c_2c_6\big(c_4 - (1-\alpha)c_2^2\big)\\
&- 30375c_3c_6\big(c_3 - (1-\alpha)c_1c_2\big)\\
&+ 56000\big(c_5 - (1-\alpha)c_2c_3\big)\big[c_3c_4 - c_2c_5\big]\\
&+ 48384\big(c_4 - (1-\alpha)c_1c_3\big)\big[c_3c_5 - c_4^2\big]\\
&+ 25600c_4c_6\big(c_2 - (1-\alpha)c_1^2\big)\\
&+ 42000(1-\alpha)\big(c_1c_5 - \frac{4}{5}c_2c_4\big)^2\\
&- 23625\big(c_3^2 - \frac{14616}{23625}c_1c_5\big)\big(c_6 - (1-\alpha)c_3^2\big)\\
&+ 17600(1-\alpha)\big(c_1c_6 - \frac{8512}{17600}c_3c_4\big)\big(c_2c_3 - c_1c_4\big)\\
&- 14616c_1c_6\big(c_5 - \frac{11717(1-\alpha)}{14616}c_2c_3\big)\\
&+ 12308(1-\alpha)c_2c_3\big(c_1c_6 - \frac{11088}{12308}c_3c_4\big)\\
&- 3584c_3c_4\big(c_5 - (1-\alpha)c_1c_4\big)\big]. \qquad (3.2)
\end{aligned}$$

Taking modulus on both sides in (3.2), we have

$$\begin{aligned}|H_{4,1,\alpha}(f)| = \frac{(1-\alpha)^3}{6048000}\big[&32000|c_2||c_6|\big|c_4-(1-\alpha)c_2^2\big| \\ &+30375|c_3||c_6|\big|c_3-(1-\alpha)c_1c_2\big| \\ &+56000\big|c_5-(1-\alpha)c_2c_3\big|\big|c_3c_4-c_2c_5\big| \\ &+48384\big|c_4-(1-\alpha)c_1c_3\big|\big|c_3c_5-c_4^2\big| \\ &+25600|c_4||c_6|\big|c_2-(1-\alpha)c_1^2\big| \\ &+42000(1-\alpha)\big|c_1c_5-\frac{4}{5}c_2c_4\big|^2 \\ &+23625\big|c_3^2-\frac{14616}{23625}c_1c_5\big|\big|c_6-(1-\alpha)c_3^2\big| \\ &+17600(1-\alpha)\big|c_1c_6-\frac{8512}{17600}c_3c_4\big|\big|c_2c_3-c_1c_4\big| \\ &+14616|c_1||c_6|\big|c_5-\frac{11717(1-\alpha)}{14616}c_2c_3\big| \\ &+12308(1-\alpha)|c_2||c_3|\big|c_1c_6-\frac{11088}{12308}c_3c_4\big| \\ &+3584|c_3||c_4|\big|c_5-(1-\alpha)c_1c_4\big|\big]. \end{aligned} \tag{3.3}$$

Case 1: When $\alpha \in \big[0, \frac{1827}{12800}\big] \cup \big[\frac{10973}{12800}, 1\big)$

Applying Lemmas 1.3, 1.4 and 1.5, a simple calculation yeilds

$$|H_{4,1,\alpha}(f)| \le \Big[\frac{(1-\alpha)^3\big(-17977\alpha+47250\big)}{94500}\Big]. \tag{3.4}$$

Case 2: When $\alpha \in \big[\frac{1827}{12800}, \frac{1}{2}\big]$

Applying Lemmas 1.3, 1.4, 1.5 and 1.6 in (3.3) and using $|c_1| = x$, upon simplification, it simplifies to

$$|H_{4,1,\alpha}(f)| \le \Big[\frac{(1-\alpha)^3\big(-3200\alpha x^2+1827x+90846-35954\alpha\big)}{189000}\Big]. \tag{3.5}$$

Now,

$$F_1(x) = \big[-3200\alpha x^2+1827x+90846-35954\alpha\big] \tag{3.6}$$

We observe that the function in (3.6) attains its maximum at $x = \frac{1827}{6400\alpha}$, so for $x = \frac{1827}{6400\alpha}$ in $F_1(x)$, upon simplification, we obtain

$$F_1 \max = F_1\Big(\frac{1827}{6400\alpha}\Big) = -35954\alpha + \frac{3337929}{12800\alpha} + 90846. \tag{3.7}$$

From (3.5) and (3.7) upon simplification, we get

$$|H_{4,1,\alpha}(f)| \leq \Big[\frac{(1-\alpha)^3\big(-460211200\alpha^2 + 3337929 + 1162828800\alpha\big)}{2419200000\alpha}\Big]. \tag{3.8}$$

Case 3: When $\alpha \in \big[\frac{1}{2}, \frac{10973}{12800}\big]$

Applying Lemmas 1.3, 1.4, 1.5 and 1.6 in (3.3) and using $|c_1| = x$, upon simplification, it simplifies to

$$|H_{4,1,\alpha}(f)| \leq \frac{(1-\alpha)^3\big(-3200x^2 + 1827x - 35954\alpha + 3200\alpha x^2 + 90846\big)}{18900}. \tag{3.9}$$

Now,

$$F_2(x) = \big(-3200x^2 + 1827x - 35954\alpha + 3200\alpha x^2 + 90846\big). \tag{3.10}$$

We observe that the function in (3.10) attains its maximum at $x = \frac{1827}{6400(1-\alpha)}$, so for $x = \frac{1827}{6400(1-\alpha)}$ in $F_2(x)$, after simplification, we obtained

$$\begin{aligned} F_2 \max &= F_2\Big(\frac{1827}{6400(1-\alpha)}\Big) \\ &= \Big[\frac{-460211200\alpha^2 + 460211200\alpha - 3337929}{12800\alpha - 12800} + 90846\Big]. \end{aligned} \tag{3.11}$$

From (3.9) and (3.11) upon simplification, we get

$$|H_{4,1,\alpha}(f)| \leq \Big[\frac{(1-\alpha)^2\big(460211200\alpha^2 - 1623040000\alpha + 1166166729\big)}{2419200000}\Big]. \tag{3.12}$$

Hence theorem. □

4. Bound of $|H_{4,1,\alpha}(f)|$ for the Set $\Re^{(m)}(\alpha)$

If a rotation of the unit disc, $\mathcal{U}_d$ about the origin through an angle $\frac{2\pi}{m}$ carries $\mathcal{U}_d$ on it self, it is said to be m-fold symmetric, where $m \in \mathbb{N}$. In $\mathcal{U}_d$, it is obvious that holomorphic mapping f is m-fold symmetric, if for each $z \in \mathcal{U}_d$

$$f(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} f(z).$$

The class of all m-fold ultivalent functions is denoted by $\mathcal{S}^m$, for which every member has the Taylor series expansion namely,

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \tag{4.1}$$

The subclass $\Re^{(m)}(\alpha)$ of $\mathcal{S}^{(m)}$ is the collection of m-fold symmetric funtions with bounded turning. It is clear that, for a holomorphic function f given in (4.1), a member of the class $\Re^{(m)}(\alpha)$ if and only if

$$f'(z) - \alpha = (1-\alpha)g(z), \text{with } g \in \mathcal{P}^{(m)}, \tag{4.2}$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{g \in \mathcal{P}^{(m)} : g(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk},\ (z \in \mathcal{U}_d)\right\}. \tag{4.3}$$

Theorem 4.1. *If $f \in \Re^{(2)}$, then $|H_{3,3}(f)| \leq \frac{8(1-\alpha)^3}{105}$.*

Proof. Let $f \in \Re^{(2)}(\alpha)$. Then there exist a function $g \in \mathcal{P}^{(2)}$ such that

$$f'(z) - \alpha = (1-\alpha)g(z). \tag{4.4}$$

From (4.1), (4.2) and (4.4), a simple calculation results

$$a_{2n+1} = \frac{(1-\alpha)c_{2n}}{2n+1}, \text{ for } n \in \{1, 2, 3\}. \tag{4.5}$$

Expanding the determinant namely $H_{3,3,\alpha}(f)$ given in (1.4) with $a_2 = a_4 = a_6 = 0$, we obtain

$$H_{3,3,\alpha}(f) = (1-\alpha)^3 a_5 (a_3 a_7 - a_5^2). \tag{4.6}$$

Substituting the values of $a_3,\ a_5$ and a_7 from (4.5), in the expression (4.6), it simplifies to give

$$H_{3,3,\alpha}(f) = \frac{(1-\alpha)^3 c_4}{105}\left[c_2c_6 - \frac{21}{25}c_4^2\right]. \tag{4.7}$$

Taking modulus on both sides in the expression (4.7), we have

$$\left|H_{3,3,\alpha}(f)\right| \leq \frac{(1-\alpha)^3|c_4|}{105}\left|c_2c_6 - \frac{21}{25}c_4^2\right|. \tag{4.8}$$

Applying the Lemma 1.4 and 1.5 in (4.8), we obtain

$$\left|H_{3,3,\alpha}(f)\right| \leq \left[\frac{8(1-\alpha)^3}{105}\right]. \tag{4.9}$$

$\square$

THEOREM 4.2. *If $f \in \Re^{(2)}(\alpha),\ \alpha \in [0,1),$ then*

$$|H_{4,1,\alpha}(f)| \leq \begin{cases} \left[\frac{4\,(1-\alpha)^3\,(2-20\alpha)}{945}\right],\ \alpha \in \left[0, \frac{1}{10}\right] \\ \\ \left[\frac{4\,(1-\alpha)^3\,(-2+20\alpha)}{945}\right],\ \alpha \in \left[\frac{1}{10}, 1\right) \end{cases}.$$

Proof. Let $f \in \Re^{(2)}(\alpha)$, upon expanding the determinant namely $H_{4,1,\alpha}(f)$ given in (1.4) with $a_2 = a_4 = a_6 = 0$, we obtain

$$H_{4,1,\alpha}(f) = \left(a_3^2 - a_5\right)\left(a_5^2 - a_3a_7\right). \tag{4.10}$$

Substituting the values of $a_3,\ a_5$ and a_7 from (4.5), in the expression (4.10), it simplifies to

$$H_{4,1,\alpha}(f) = \frac{(1-\alpha)^3}{105}\left(c_4 - (1-\alpha)\frac{5}{9}c_2^2\right)\left(c_2c_6 - \frac{21}{25}c_4^2\right). \tag{4.11}$$

Now, in view of obvious equivalence $g \in \mathcal{P}^{(2)} \iff q \in \mathcal{P}$ provided $g(z) = q(z^2)$, we can write

$$\max\left\{\left(c_4 - (1-\alpha)\frac{5}{9}c_2^2\right) : g \in \mathcal{P}^{(2)}\right\},$$

as

$$\max\left\{\left(c_2 - (1-\alpha)\frac{5}{9}c_1^2\right) : g \in \mathcal{P}\right\}. \tag{4.12}$$

From expresssions (4.11) and (4.12), we obtained

$$H_{4,1,\alpha}(f) = \frac{(1-\alpha)^3}{105}\big(c_2 - (1-\alpha)\frac{5}{9}c_1^2\big)\big(c_2c_6 - \frac{21}{25}c_4^2\big). \tag{4.13}$$

Case-1: When $\alpha \in \big[0, \frac{1}{10}\big)$ Taking modulus on both side in (4.13), upon applying Lemmas 1.4, 1.5 and 1.6, after simplification, it simplifies to

$$\big|H_{4,1,\alpha}(f)\big| \leq \big[\frac{4\big(1-\alpha\big)^3\big(2-20\alpha\big)}{945}\big]. \tag{4.14}$$

Case-2: When $\alpha \in \big[\frac{1}{10}, 1\big)$ Taking modulus on both side in (4.13),applying Lemmas 1.4, 1.5 and 1.6, upon simplification, yeilds

$$\big|H_{4,1,\alpha}(f)\big| \leq \big[\frac{4\big(1-\alpha\big)^3\big(-2+20\alpha\big)}{945}\big]. \tag{4.15}$$

Hence the theorem. □

Theorem 4.3. *If* $f \in \Re^{(3)}(\alpha),\ \alpha \in [0,1)$, *then*

$$|H_{4,1,\alpha}(f)| \leq \big[\frac{\big(1-\alpha\big)^3\big(1+7\alpha\big)}{112}\big].$$

Proof. Let $f \in \Re^{(3)}(\alpha)$. Then there exist a function $g \in \mathcal{P}^{(m)}$ such that

$$f'(z) - \alpha = (1-\alpha)g(z). \tag{4.16}$$

From (4.1),(4.2) and (4.17), a simple calculation results

$$a_4 = \frac{(1-\alpha)c_3}{4} \text{ and } a_7 = \frac{(1-\alpha)c_6}{7}. \tag{4.17}$$

Upon expanding the determinant namely $H_{4,1,\alpha}(f)$ given in (1.4) with $a_2 = a_3 = a_5 = a_6 = 0$, we obtain

$$H_{4,1,\alpha}(f) = a_4^2\big(a_4^2 - a_7\big). \tag{4.18}$$

Substituting the values of a_4 and a_7 from (4.18), in the expression (4.19), it simplifies to

$$H_{4,1,\alpha}(f) = \frac{(1-\alpha)^3c_3^2}{16}\big(\frac{(1-\alpha)c_3^2}{16} - \frac{c_6}{7}\big). \tag{4.19}$$

Rearranging the terms in the expression (4.20) in order to apply lemmas mentioned in this paper, we have

$$H_{4,1,\alpha}(f) = -\frac{(1-\alpha)^3 c_3^2}{112}\big(c_6 - \frac{7(1-\alpha)}{16}c_3^2\big). \tag{4.20}$$

Now, in view of equivalence $g \in \mathcal{P}^{(3)} \iff q \in \mathcal{P}$ provided $g(z) = q(z^3)$, we can write

$$\max\big\{c_3^2\big(c_6 - \frac{7(1-\alpha)}{16}c_3^2\big) : g \in \mathcal{P}^{(3)}\big\},$$

as

$$\max\big\{c_1^2\big(c_2 - \frac{7(1-\alpha)}{16}c_1^2\big) : g \in \mathcal{P}\big\}. \tag{4.21}$$

From equations (4.21) and (4.22), we get

$$H_{4,1,\alpha}(f) = -\frac{(1-\alpha)^3 c_1^2}{112}\big(c_2 - \frac{7(1-\alpha)}{16}c_1^2\big). \tag{4.22}$$

Taking modulus on both side of the equation (4.23), we obtain

$$|H_{4,1,\alpha}(f)| \leq \frac{(1-\alpha)^3|c_1|^2}{112}\big|c_2 - \frac{7(1-\alpha)}{16}c_1^2\big|. \tag{4.23}$$

Upon applying the Lemmas 1.4 and 1.6 in the expression (4.24), we obtained

$$|H_{4,1,\alpha}(f)| \leq \big[\frac{\big(1-\alpha\big)^3\big(1+7\alpha\big)}{112}\big]. \tag{4.24}$$

Thus, the theorem is proved. □

5. Remarks

REMARK 5.1. For the choice of $\alpha = 0$ in (2.6), we get

$$|H_{3,3}(f)| \leq \frac{22}{105} \approx 0.209523.$$

The inequlity obtained here is more refined than that of Arif et al. [1].

REMARK 5.2. For $\alpha = 0$ in (3.4), we obtain

$$|H_{4,1}(f)| \leq \frac{1}{2}.$$

The bound obtained above is more refined than that of Arif et al. [3].

Remark 5.3. For the choice of $\alpha = 0$ in (4.9), we get

$$\left|H_{3,3}(f)\right| \leq \frac{8}{105} \approx 0.07619.$$

Remark 5.4. For the choice $\alpha = 0$ in (4.14)

$$\left|H_{4,1}(f)\right| \leq \frac{8}{945} \approx 0.008465, \tag{5.1}$$

the obtained bound is more refined than the bound developed by Arief et al. [3].

Remark 5.5. For the choice of $\alpha = 0$ in (4.25), we obtained

$$|H_{4,1}(f)| \leq \frac{1}{112} \approx 0.0089285, \tag{5.2}$$

the obtained bound is more refined than that of Arief et al. [3].

Concluding Remarks and scope of further Research: In this paper, we estimated an upper bound for the Hankel determinants of third and fourth orders for the class $R(\alpha)$, $(0 \leq \alpha < 1)$ of bounded turning functions of order α. The bounds obtained are more refined than the bounds obtained by the earlier authors, when the parameter $\alpha = 0$. Further, the bounds estimated for two-fold and three-fold symmetric functions for $|H_{4,1,\alpha}(f)|$ are more polished, for $\alpha = 0$. Finally, one can also attempt to find sharp bounds for $|H_{3,3}(f)|$, $|H_{4,1}(f)|$ the functions belonging to the same class $R(\alpha)$.

Acknowledgement

The authors are highly grateful to the esteemed Referee(s) for a comprehensive reading of the manuscript and making valuable suggestions, leading to a better paper presentation.

Conflict of Interest: The authors declare that they have no conflict of interest.

References

[1] M. Arif, Inayat Ullah, Mohsan Raza and Pawel Zaprawa, Investigation of the Fifth Hankel determinant for a family of functions with bounded turning, Math. Slovaca, **70**(2)(2020), 319-328.

[2] M. Arif, Lubna Rani, Mohsan Raza and P. Zaprawa, Fourth Hankel determinant for the set of star-like functions, Math. Probl. Eng., **2021**, Article ID 6674010, 8 pp, https://doi.org/10.1155/2021/6674010.

[3] M. Arif, Mohsan Raza, Huo Tang, Shehzad Hussain and Hassan Khan, Hankel determinant of order three for familiar subsets of analytic functions related with sine function, Open Math., **17**(1)(2019), 1615-1630.

[4] M. Arif, Lubna Rani, Mohsan Raza and P. Zaprawa, Fourth Hankel determinant for the family of functions with bounded turning, Bull. Korean Math. Soc., **55**(6)(2018), 1703-1711.

[5] K. O. Babalola, On $H_3(1)$ Hankel determinant for some classes of univalent functions, Inequality Theory and Applications, 6, ed. Y. J. Cho, Nova Sci. Publ., New York, 2010, 1-7.

[6] D. Bansal, S. Maharana and J. K. Prajapat, Third order Hankel determinant for certain univalent functions, J. Korean Math. Soc., **52**(6)(2015), 1139-1148.

[7] S Banga and S. Sivaprasad Kumar, The sharp bounds of the second and third Hankel determinants for the class SL*, Math. Slovaca, **70**(4)(2020), 849-862, DOI: 10.1515/ms-2017-0398.

[8] P. L. Duren, Univalent functions, **259** of Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, 1983.

[9] Gaganpreet Kaur, Gurmeet Singh, Muhammad Arif, Ronnason Chinram and Javed Iqbal, A study of third and fourth Hankel determinant problem for a particular class of bounded turning functions, Mathematical Problems in Engineering, **2021**, Article ID 6687805, https://doi.org/10.1155/2021/6687805.

[10] T. Hayami and S. Owa, Generalized Hankel determinant for certain classes, Int. J. Math. Anal., **4**(52)(2010), 2573-2585.

[11] A. Janteng, S. A. Halim and M. Darus, Hankel determinant for starlike and convex functions, Int. J. Math. Anal., **1**(13)(2007), 619-625.

[12] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, J. Inequal. Pure Appl. Math., **7**(2)(2006), 1-5.

[13] A. Lecko, Young Jae Sim and Barabara Smiarowska, The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2, Complex Anal. Oper. Theory, **13**(5)(2019), 2231-2238.

[14] B. Kowalczyk, Adam Lecko and Young Jae Sim, The sharp bound for the Hankel determinant of the third kind for convex functions, Bull. Aust. Math. Soc., **97**(3)(2018), 435-445.

[15] O. S. Kwon, A. Lecko and Y. J. Sim, The bound of the Hankel determinant of the third kind for starlike functions, Bull. Malays. Math. Sci. Soc., **42**(2)(2019), 767-780.

[16] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in $\mathscr{P}$, Proc. Amer. Math. Soc., **87**(2)(1983), 251-257.

[17] A. E. Livingston, The coefficients of multivalent close-to-convex functions, Proc. Amer. Math. Soc., **21**(3)(1969), 545-552.

[18] T. H. MacGregor, Functions whose derivative have a positive real part, Trans. Amer. Math. Soc., **104**(3)(1962), 532-537.

[19] Milutin Obradovic and Nikola Tuneski, New upper bounds of the third Hankel determinant for some classes of univalent functions, arXiv:1911.10770.

[20] H. Orhan and P. Zaprawa, Third Hankel determinants for starlike and convex functions of order alpha, Bull. Korean Math. Soc., **55**(1)(2018), 165-173.

[21] S. M. Patil and S. M. Khairnar, Third Hankel determinant for starlike function with respect to symmetric points, Int. J. Pure Appl. Math. Sci., **10**(1)(2017), 7-15.

[22] Ch. Pommerenke, Univalent Functions, Gottingen, Vandenhoeck and Ruprecht, 1975.

[23] Ch. Pommerenke, On the coefficients and Hankel determinants of univalent functions, J. Lond. Math. Soc., **41**(s-1)(1966), 111-122.

[24] J. K. Prajapat, D. Bansal, A. Singh and A. K. Mishra, Bounds on third Hankel determinant for close-to-convex functions, Acta Univ. Sapientiae, Mathematica, Acta Univ. Sapientiae Math., **7**(2), DOI:10.1515/ausm-2015-0014.

[25] B. Rath, K. S. Kumar, D. V. Krishna and A. Lecko, The sharp bound of the third Hankel determinant for starlike functions of order 1/2, Complex Anal. Oper. Theory, (2022), https://doi.org/10.1007/s11785-022-01241-8.

[26] M. Raza and S. N. Malik, Upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli, J Inequal. Appl. (2013). Art. 412. doi:10.1186/1029-242X-2013-412.

[27] L Shi, Izaz Ali, Muhammad Arif, Nak Eun Cho, Shehzad Hussain and Hassan Khan, A study of third Hankel determinant problem for certain subfamilies of analytic functions involving cardioid domain, Mathematics, **7**(5)(2019), 418, https://doi.org/10.3390/math7050418.

[28] Y. J. Sim and Pawel Zaprawa, Third Hankel determinants for two classes of analytic functions with real coefficients, Forum Math., **33**(4)(2021), 973-98, https://doi.org/10.1515/forum-2021-0014.

[29] H. M. Srivastavaa, Bilal Khan, Nazar Khan, Muhammad Tahir, Sarfraz Ahmad and NasirKhan, Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with the q-exponential function, Bull. Sci. Math., **167**(2021), https://doi.org/10.1016/j.bulsci.2020.102942.

[30] T. V. Sudharsan, S. P. Vijayalakshmi and B. Adolf Stephen, Third Hankel determinant for a subclass of analytic functions, Malaya J. Math., **2**(4)(2014), 438-444.

[31] D. Vamshee Krishna and T. RamReddy, Coefficient inequality for certain p- valent analytic functions, Rocky Mountain J. Math., **44**(6)(2014), 1941-1959.

[32] P. Zaprawa, Milutin Obradovic and Nikola Tuneski, Third Hankel determinant for univalent starlike functions, Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemticas, **115**(2021), Article number:49.

[33] P. Zaprawa, On Hankel determinant $H_2(3)$ for univalent functions, Results Math., **73**(3)(2018), article 89, 1-12.

[34] P. Zaprawa, Third Hankel determinants for subclasses of univalent functions, Mediterr. J. Math., **14**(1)(2017), 1-10.

Department of Mathematics
Gitam Institute of Science
GITAM University
Visakhapatnam- 530 045
Andhra Pradesh
India.

E-mail: brath@gitam.edu
skarri9@gitam.in
vamsheekrishna1972@gmail.com
vnallmot@gitam.in

Department of Mathematics
Sri Gurajada Apparao Government Degree College
Yalamanchili- 531 055
Andhra Pradesh
India.

E-mail: vijay.chalumuri123@gmail.com