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The sharp estimates for certain coefficient inequalities for transforms of a starlike function associated with Bernoulli's lemniscate

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This paper contains an estimation of sharp upper bound (UB) for the Hankel determinant of specific orders concerning the n^{th}-root transform $[f(z^n)]^{\frac{1}{n}}$ for a subfamily of holomorphic functions associated with the right half of Bernoulli's lemniscate. The practical tools applied in the derivation of our main results are the coefficient inequalities of the Carathéodory class $\mathcal{P}$.

Keywords: Holomorphic starlike function; subordination; lemniscate of Bernoulli; Hankel determinant.

AMS Subject Classification: 30C45, 30C50

1. Introduction

Denote the collection of holomorphic mappings f of the type

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

in the open unit disk $\mathcal{U}_d = \{z \in \mathbb{C} : 1 > |z|\}$, satisfying the normalized conditions namely $f(0) = f'(0) - 1 = 0$ as $\mathcal{A}$ and S is a subfamily of $\mathcal{A}$, which contains the

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univalent (schlicht) mappings. For two analytic functions F and G in the unit disk $\mathcal{U}_d$, F is said to be subordinate to G, written $F \prec G$, if there exists a regular function $w(z)$ in the open unit disk $\mathcal{U}_d$ with $w(0) = 1$ and $|w(z)| < 1$, $z \in \mathcal{U}_d$, called Schwarz's function such that

$$F(z) = G(w(z)), \quad z \in \mathcal{U}_d.$$

If $F \prec G$ and $G(z)$ is univalent in the open unit disk $\mathcal{U}_d$, then the subordination is equivalent to $F(0) = G(0)$ and range $F(z) \subseteq$ range $G(z)$. The n^{th}-root transform of f given in (1.1) is

$$G(z) := [f(z^n)]^{\frac{1}{n}} = z + \sum_{s=1}^{\infty} b_{ns+1} z^{ns+1}. \tag{1.2}$$

At this sequel, we have launched and interpreted the concept of Hankel determinant of $G(z)$ for the function f in (1.1), with $q, t, n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ as

$$H_{q,t,n}(f) = \begin{vmatrix} b_{n(t-1)+1} & b_{nt+1} & \cdots & b_{n(t+q-2)+1} \\ b_{nt+1} & b_{n(t+1)+1} & \cdots & b_{n(t+q-1)+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n(t+q-2)+1} & b_{n(t+q-1)+1} & \cdots & b_{n[t+2(q-1)-1]+1} \end{vmatrix} \quad (b_1 = 1). \tag{1.3}$$

For $n = 1$ in (1.3), it reduces to $H_{q,t,n}(f) = H_{q,t,1}(f) = H_{q,t}(f)$, this Hankel determinant was defined by Pommerenke [10] for f given in (1.1). Specifically for $q = 2, t \in \{1, 2\}$, the Hankel determinant in (1.3) has been simplified, respectively, to

$$H_{2,1,n}(f) = \begin{vmatrix} 1 & b_{n+1} \\ b_{n+1} & b_{2n+1} \end{vmatrix} = b_{2n+1} - b_{n+1}^2, \tag{1.4}$$

$$H_{2,2,n}(f) = \begin{vmatrix} b_{n+1} & b_{2n+1} \\ b_{2n+1} & b_{3n+1} \end{vmatrix} = b_{n+1}b_{3n+1} - b_{2n+1}^2. \tag{1.5}$$

Ali *et al.* [1, 2] derived exact estimates for $|b_{2n+1} - \mu b_{n+1}^2|$, which represents the generalized Fekete–Szegö functional related to n^{th}-root transforms for f in (1.1), when it is a member of certain subclasses of S. We mention $H_{2,2,n}(f)$ as the second-order Hankel determinant for the function f as mentioned in (1.2). Janteng *et al.* [6, 7] estimated sharp upper bounds (UBs) to $|H_{2,2,1}(f)| = |a_2a_4 - a_3^2|$ obtained for $n = 1$ in (1.5), for the families of bounded turning, starlike and convex functions, symbolized as $\Re$, S^* and $\mathcal{K}$, respectively, fulfilling the analytic conditions $\text{Re}\{f'(z)\} > 0$, $\text{Re}\{\frac{zf'(z)}{f(z)}\} > 0$ and $\text{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0$ in the unit disk $\mathcal{U}_d$, as $4/9$, 1 and $1/8$. With the motivation by the results obtained by different authors (see [4, 9, 13, 16, 18]), in this paper, we are making an attempt to derive sharp UB for the functionals $H_{2,1,n}(f)$ and $H_{2,2,n}(f)$ for the class of starlike functions associated with the lemniscate of Bernoulli, denoted by $\mathcal{SL}^*$, whose polar coordinates form is $r^2 = 2\cos(2\theta)$, a subfamily of $\mathcal{A}$, is defined as follows.

Definition 1.1. A mapping $f \in \mathcal{A}$ is a member of the class $\mathcal{SL}^*$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z} = q(z), \quad z \in \mathcal{U}_d,$$

where the square root mapping is considered at principal branch, that is,

$$\sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \frac{z^3}{16} - \frac{5z^4}{128} + \cdots \bigg|_{z=0} = 1.$$

It is noticed that the collection $q(\mathcal{U}_d)$ is accommodated in Bernoulli's lemniscate, whose Cartesian form is $(x^2+y^2)^2 - 2(x^2-y^2) = 0$. Therefore,

$$\mathcal{SL}^* = \left\{ f \in \mathcal{A} \colon \left| \left\{ \frac{zf'(z)}{f(z)} \right\}^2 - 1 \right| < 1 \right\}, \quad z \in \mathcal{U}_d.$$

The family $\mathcal{SL}^*$ was put forward and labored by Sokol and Stankiewicz [15], and later this class was studied by some authors (see [12, 14, 17]).

The essential tools in establishing our main results are the sharp estimates for the class $\mathcal{P}$, of all functions g, each one called as Carathéodory function [3] of the form

$$g(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \tag{1.6}$$

holomorphic in $\mathcal{U}_d$, whose real part is positive, given in the form of lemmas as follows.

Lemma 1.2 ([11]). *For $g \in \mathcal{P}$, then $|c_k| \leq 2$, for $k \in \mathbb{N}$, the inequality is sharp for the function $h(z) = \frac{1+z}{1-z}$, $z \in \mathcal{U}_d$.*

Lemma 1.3 ([5]). *If $g \in \mathcal{P}$, then the following estimate holds for the positive integers k, l, with $k > l$:*

$$|c_k - \mu c_l c_{k-l}| \leq \begin{cases} 2 & \text{if } \mu \in [0,1], \\ 2|2\mu - 1| & \text{if } \mu > 1. \end{cases}$$

Lemma 1.4 ([8]). *Let $g \in \mathcal{P}$ of the form in (1.6) with $c_1 \geq 0$ then*

$$2c_2 = c_1^2 + y(4 - c_1^2) \quad \text{and}$$
$$4c_3 = [c_1^3 + \{2c_1 y - c_1 y^2 + 2(1-|x|^2)y\}(4 - c_1^2)],$$

for some x, y such that $|x| \leq 1$ and $|y| \leq 1$.

We apply the method developed by Libera and Zlotkiewicz [8] in obtaining our results.

2. Main Results

Theorem 2.1. *If $f \in \mathcal{SL}^*$ and G, mentioned in (1.2) is the n^{th}-root transformation of f then*

$$|H_{2,1,n}(f)| = |b_{2n+1} - b_{n+1}^2| \leq \frac{1}{4n},$$

the inequality is sharp for the function

$$f(z) = z \exp \int_0^z \left[\frac{(1+t^2)^{\frac{1}{2}} - 1}{t}\right] dt = z + \frac{z^3}{4} + \cdots.$$

Proof. Let $f \in \mathcal{SL}^*$, as per Definition 1.1, we have

$$\frac{zf'(z)}{f(z)} \prec q(z) = \sqrt{1+z}. \tag{2.1}$$

According to the principle of subordination, there occurs Schwarz's mapping $\phi(z)$ such that $\phi(0) = 0$ with $|\phi(z)| < 1$ for $z \in \mathcal{U}_d$, we have

$$\frac{zf'(z)}{f(z)} = q\{\phi(z)\}, \quad z \in \mathcal{U}_d. \tag{2.2}$$

Let us take the mapping

$$\phi(z) = \frac{g(z)-1}{g(z)+1} \Leftrightarrow g(z) = \frac{1+\phi(z)}{1-\phi(z)} = 1 + \sum_{i=1}^{\infty} c_i z^i. \tag{2.3}$$

From (2.1)–(2.3), we derive

$$\frac{zf'(z)}{f(z)} = q\{\phi(z)\} = \sqrt{1 + \left(\frac{g(z)-1}{g(z)+1}\right)} = \sqrt{\frac{2g(z)}{g(z)+1}}. \tag{2.4}$$

Substituting the values for f and g in (2.4), a simple calculation gives

$$\begin{aligned}
&[1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 \\
&\quad + (4a_5 - 4a_2a_4 - 2a_3^2 + 4a_3a_2^2 - a_2^4)z^4 + \cdots] \\
&= \left[1 + \frac{1}{4}c_1 z + \left\{\frac{1}{4}c_2 - \frac{5}{32}c_1^2\right\} z^2 + \left\{\frac{1}{4}c_3 - \frac{5}{16}c_1c_2 + \frac{13}{128}c_1^3\right\} z^3\right. \\
&\quad \left. + \left\{\frac{1}{4}c_4 - \frac{5}{32}c_2^2 - \frac{5}{16}c_1c_3 + \frac{39}{128}c_1^2c_2 - \frac{141}{2048}c_1^4\right\} z^4 + \cdots\right].
\end{aligned} \tag{2.5}$$

Upon equating the coefficients of z^i, for $i \in \{1,2,3,4\}$ in (2.5), after simplifying, we get

$$\begin{aligned}
&a_2 = \frac{c_1}{4}; \quad a_3 = \frac{1}{8}\left(c_2 - \frac{3}{8}c_1^2\right); \quad a_4 = \frac{1}{48}\left(4c_3 - \frac{7}{2}c_1c_2 + \frac{13}{16}c_1^3\right); \\
&a_5 = \frac{1}{96}\left(6c_4 - 3c_2^2 - \frac{11}{2}c_1c_3 + \frac{17}{4}c_1^2c_2 - \frac{49}{64}c_1^4\right).
\end{aligned} \tag{2.6}$$

For the mapping f in (1.1), a calculation gives

$$\begin{aligned}
[f(z^n)]^{\frac{1}{n}} = \left[z^n + \sum_{n=2}^{\infty} a_n z^n\right]^{\frac{1}{n}} &= \left[z + \frac{1}{n}a_2 z^{n+1} + \left\{\frac{1}{n}a_3 + \frac{(1-n)}{2n^2}a_2^2\right\} z^{2n+1}\right.\\
&\quad + \left\{\frac{1}{n}a_4 + \frac{(1-n)}{n^2}a_2a_3 + \frac{(1-n)(1-2n)}{6n^3}a_2^3\right\} z^{3n+1}\\
&\quad + \left\{\frac{1}{n}a_5 + \frac{(1-n)}{2n^2}(a_3^2 + 2a_2a_4) + \frac{(1-n)(1-2n)}{2n^3}a_2^2a_3\right.\\
&\quad \left.\left. + \frac{(1-n)(1-2n)(1-3n)}{24n^4}a_2^4\right\} z^{4n+1} + \cdots\right].
\end{aligned} \tag{2.7}$$

Equating the coefficients of z^{n+1}, z^{2n+1}, z^{3n+1} and z^{4n+1} in the expressions (1.2) and (2.7), we obtain

$$\begin{aligned}
b_{n+1} &= \frac{1}{n}a_2;\\
b_{2n+1} &= \frac{1}{n}a_3 + \frac{(1-n)}{2n^2}a_2^2;\\
b_{3n+1} &= \left[\frac{1}{n}a_4 + \frac{(1-n)}{n^2}a_2a_3 + \frac{(1-n)(1-2n)}{6n^3}a_2^3\right];\\
b_{4n+1} &= \left[\frac{1}{n}a_5 + \frac{(1-n)}{2n^2}(a_3^2 + 2a_2a_4) + \frac{(1-n)(1-2n)}{2n^3}a_2^2a_3\right.\\
&\qquad \left. + \frac{(1-n)(1-2n)(1-3n)}{24n^4}a_2^4\right].
\end{aligned} \tag{2.8}$$

From the expressions (2.6) and (2.8), upon simplification, we obtain

$$\begin{aligned}
b_{n+1} &= \frac{c_1}{4n}; \quad b_{2n+1} = \frac{c_2}{8n} - \frac{(5n-2)}{64n^2}c_1^2;\\
b_{3n+1} &= \left[\frac{c_3}{12n} - \frac{(10n-3)}{96n^2}c_1c_2 + \frac{(26n^2-15n+2)}{768n^3}c_1^3\right];\\
b_{4n+1} &= \left[\frac{c_4}{16n} - \frac{(5n-1)}{128n^2}c_2^2 - \frac{(15n-4)}{192n^2}c_1c_3 + \frac{(117n^2-55n+6)}{1536n^3}c_1^2c_2\right.\\
&\qquad \left. - \frac{(9n-2)(47n^2-21n+2)}{24576n^4}c_1^4\right].
\end{aligned} \tag{2.9}$$

Substituting the calculated values of b_{n+1} and b_{2n+1} from (2.9) in $H_{2,1,n}(f)$, it takes the form

$$H_{2,1,n}(f) = b_{2n+1} - b_{n+1}^2 = \frac{1}{8n}\left[c_2 - \left(\frac{5n+2}{8n}\right)c_1^2\right]. \tag{2.10}$$

Take the modulus on both sides and then apply Lemma 1.3 in (2.10), which follows the result of Theorem 2.1. $\square$

Theorem 2.2. *For the n^{th}-root transformation of f namely G, specified in* (1.2) *and if $f \in \mathcal{SL}^*$ then*

$$|H_{2,2,n}(f)| = |b_{n+1}b_{3n+1} - b_{2n+1}^2| \leq \frac{1}{16n^2}$$

and the inequality is sharp for the same function given in Theorem 2.1.

Proof. Putting the calculated values of b_{jn+1}, for $j \in \{1, 2, 3\}$ from (2.9) in the expression $b_{n+1}b_{3n+1} - b_{2n+1}^2$, it simplifies to

$$\begin{aligned} &b_{n+1}b_{3n+1} - b_{2n+1}^2 \\ &\quad = \frac{1}{12288n^4} \times [256n^2c_1c_3 - 80n^2c_1^2c_2 - 192n^2c_2^2 + (29n^2-4)c_1^4], \end{aligned} \tag{2.11}$$

which is equivalent to

$$b_{n+1}b_{3n+1} - b_{2n+1}^2 = \frac{1}{12288n^4} \times [d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4], \tag{2.12}$$

where

$$d_1 = 256n^2; \quad d_2 = -80n^2; \quad d_3 = -192n^2; \quad d_4 = (29n^2-4). \tag{2.13}$$

Substituting the values of c_2 and c_3 from Lemma 1.4 on the right-hand side of the expression (2.12), it simplifies to

$$\begin{aligned} &4[d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4] \\ &\quad = [(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + 2(d_1 + d_2 + d_3)c_1^2(4-c_1^2)x \\ &\qquad - d_1c_1^2(4-c_1^2)x^2 + d_3(4-c_1^2)^2x^2 + 2d_1c_1(4-c_1^2)(1-|x|^2)y]. \end{aligned} \tag{2.14}$$

Taking modulus on both sides and then applying the triangle inequality in the above expression, we have

$$\begin{aligned} &4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \\ &\quad \leq [|d_1 + 2d_2 + d_3 + 4d_4||c_1|^4 + 2|d_1 + d_2 + d_3||c_1|^2|4-c_1^2||x| \\ &\qquad + \{(|d_1| - |d_3)c_1^2 - 2|d_1||c_1||y| + 4|d_3|\}|4-c_1^2||x|^2 \\ &\qquad + 2|d_1||c_1||4-c_1^2||d_1|. \end{aligned} \tag{2.15}$$

From (2.13), we can now write

$$\begin{aligned} &|d_1 + 2d_2 + d_3 + 4d_4| = 4(5n^2-4); \quad |d_1 + d_2 + d_3| = 16n^2; \\ &\{(|d_1| - |d_3)c_1^2 - 2|d_1||c_1||y| + 4|d_3|\} = 64n^2(c_1^2 - 8c_1|y| + 12). \end{aligned} \tag{2.16}$$

Putting the calculated values from (2.16) and the value of d_1 from (2.13) in (2.15), after simplifying, we get

$$\begin{aligned}
&|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \\
&\quad \leq [(5n^2-4)c_1^4 + 128n^2c_1(4-c_1^2)|y| + 8n^2c_1(4-c_1^2)|x| \\
&\quad\quad + 16n^2(c_1^2 - 8c_1|y| + 12)(4-c_1^2)|x|^2]. \qquad (2.17)
\end{aligned}$$

Applying the triangle inequality, restoring $|x|$ by ρ, with $|y| \leq 1$, and choosing $c_1 = c \in [0,2]$, on the right-hand side of (2.17), we obtain

$$\begin{aligned}
&|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \\
&\quad \leq [(5n^2-4)c^4 + 128n^2c(4-c^2) + 8n^2c^2(4-c^2)\rho \\
&\quad\quad + 16n^2(c-2)(c-6)(4-c^2)\rho^2] \\
&\quad = H(c,\rho) \quad \text{for } |x| = \rho \in [0,1]. \qquad (2.18)
\end{aligned}$$

Here,

$$H(c,\rho) = (5n^2-4)c^4 + 8n^2\{16c + c^2\rho + 2(c-2)(c-6)\rho^2\}(4-c^2). \qquad (2.19)$$

To determine the maximum value of $H(c,\rho)$ over the region of the rectangle $[0,1] \times [0,2]$, we consider the partial differential coefficient of $H(c,\rho)$, from (2.19) with regard to ρ, given by

$$\frac{\partial H}{\partial \rho} = 8n^2[c^2 + 4(c-2)(c-6)\rho](4-c^2). \qquad (2.20)$$

For $\rho \in (0,1)$ and fixed $c \in (0,2)$, from (2.20), we notice that $\frac{\partial H}{\partial \rho} > 0$, which indicates that $H(c,\rho)$ turns out to be an increasing mapping of ρ, whose highest value takes place on the border only, i.e. when $\rho = 1$. Hence, for $\rho = 1$ in (2.19), after simplifying, we get

$$F(c) = H(c,1) = -(19n^2+4)c^4 - 32n^2c^2 + 768n^2, \qquad (2.21)$$

$$F'(c) = -4(19n^2+4)c^3 - 64n^2c, \qquad (2.22)$$

$$F''(c) = -12(19n^2+4)c^2 - 64n^2. \qquad (2.23)$$

For the extreme values of $F(c)$, let $F'(c) = 0$. From (2.22), we have

$$-4c\{(19n^2+4)c^2 + 16n^2\} = 0. \qquad (2.24)$$

Now, let us discuss the following two instances.

Case (1). When $c = 0$, from (2.23), we observe that

$$F''(c) = -64n^2 < 0 \quad \text{for } n \in \mathbb{N}.$$

Therefore, by the second differentiation test at $c = 0$, $F(c)$ possesses the most significant value, which we can obtain from (2.21) as

$$\max_{0 \le c \le 2} F(0) = 768n^2. \tag{2.25}$$

Case (2). When $c \neq 0$, from (2.24), we get

$$c^2 = \left\{ \frac{-16n^2}{19n^2 + 4} \right\}. \tag{2.26}$$

Since $n \in \mathbb{N}$, from (2.26), we observe that, c^2 is not in $[0, 2]$.

Now, simplifying the expressions (2.18) and (2.25), in view of (2.12), we obtain

$$|b_{n+1}b_{3n+1} - b_{2n+1}^2| = |d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \le \frac{1}{16n^2}. \tag{2.27}$$

This completes the proof of our theorem. □

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